Appl. Phys. B 45, 37–44 (1988)

Applied Physics B Photophysics and Laser Chemistry

Analysis of Raman Gain for Focussed Gaussian Pump Beams

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Received 13 May 1987/Accepted 9 September 1987

Abstract. Several theoretical and numerical models have been published which describe the evolution of a Stokes beam in a Raman medium excited by a focussed pump beam. Generally, the published theoretical departures from the plane-wave theory of Raman scattering are based on assumptions about the power of the pump beam. In this paper we present a theoretical model which is shown to be in excellent agreement with an exact numerical treatment, and which is valid without restrictions on the pump power. Its predictions are used to indicate the range of validity of earlier theories.

PACS: 42.65, 78.30j

Several authors have addressed the problem of describing the spatial behaviour of a Stokes field in a Raman active medium driven by a focussed pump beam [1–7] and [13]. The motivation is generally to discover the optimum experimental conditions required to effect an efficient conversion of energy from the pump to the Stokes field. The information required may include the necessary pump power such that the Stokes field attain some threshold, the optimum focussing conditions for the pump beam that minimise the threshold, and the Stokes beam parameters on leaving the medium.

In this paper, we present an approximate solution to the problem using a variational technique for describing the evolution of the Stokes field in the gain medium as a Gaussian (TEM_{00}) beam. This solution is shown to be in excellent agreement with an exact numerical treatment of Perry et al. [5, 6], and that the work of earlier researchers [1–4] are special cases of the general results we derive, each having associated limited domains of applicability. Thus the "overlap integral" approach of Boyd et al. [1] is seen to be the limit for low pump power in our more general result, whilst the solution developed by Cotter et al. [2] based upon a quadratic index profile approximation is shown to be applicable only when the pump power is sufficiently large.

The following section deals with the derivation of the equations of motion for the parameters describing the Stokes field. The next section presents the solutions to these equations under the high and low pump power limits discussed above followed by the more general result of our analysis. The final section is a discussion of the predictions of this result together with a comparison with the results of Perry et al. [5, 6].

1. Equations of Motion for the Stokes Field

In the following analysis we assume that the Stokes field growth is small signal, steady-state, and without

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competing processes. The conditions to be satisfied are respectively:

(a) The intensity of the Stokes field is not large enough to deplete the pump or saturate the medium.

(b) The pump and Stokes field each have a bandwidth smaller than the Raman linewidth [8].

(c) The gain and material dispersion of the medium favours the dominant growth of a field at the first Stokes frequency over higher-order Raman processes [9].

Our starting point in the variational approach to the derivation of the Stokes field is the Lagrangian density for the electromagnetic field [10]

$$\mathscr{L}=(1/2)(\mathbf{D}\cdot\mathbf{E}-\mathbf{B}\cdot\mathbf{H}) \tag{1}$$

and the Maxwell relation

$$\partial\mathbf{H}/\partial t=(1/\mu_0)\nabla\times\mathbf{E}. \tag{2}$$

The pump and Stokes fields are defined as those components of the total field with frequencies ω_p and ω_s, respectively. In the small signal regime, the pump field is unperturbed by the medium and its spatial distribution may therefore be regarded as given. Thus (1 and 2) apply to the field components at the Stokes frequency only, which we expand in the usual manner making explicit the rapidly varying part of the spatial variation in the z direction

$$\mathbf{E}=\mathrm{Re}\{e_s(\mathbf{r})\exp[\mathrm{i}(\omega_s t-k_s z)]\hat{\mathbf{e}}_s\} \tag{3}$$

and similarly for the magnetic field. Here $\hat{\mathbf{e}}_s$ is a unit vector in the direction of polarisation of the Stokes field; $k_s=n_s\omega_s/c$, where n_s is the refracfive index at the Stokes frequency; and $e_s(\mathbf{r})$ is a slowly varying envelope. In addition to (a–c), it is assumed in this paper that the pump field is a Gaussian beam and that the Stokes field is linearly polarised parallel to the pump field. Then the fields $\mathbf{D}$ and $\mathbf{B}$ can be written in terms of the electric and magnetic field vectors as follows

$$\mathbf{B}=\mu_0\mathbf{H}; \quad \mathbf{D}=\varepsilon_0 n_s^2\mathbf{E}+\mathbf{P}, \tag{4}$$

$$\mathbf{P}=(3/2)\varepsilon_0|e_p(\mathbf{r})|^2\chi^{(R)}\mathbf{E}, \tag{5}$$

$$|e_p(\mathbf{r})|^2=|e_{p0}|^2(w_{p0}/w_p(z))^2\exp[-2(r/w_p(z))^2], \tag{6}$$

$$w_p^2(z)=w_{p0}^2[1+(2z/k_p w_{p0}^2)^2], \tag{7}$$

where $\chi^{(R)}$ is the Raman susceptibility, the definition of which is taken from Hanna et al. [11]. Classically, the Stokes field $e_s(\mathbf{r})$ will be that distribution for which the integral of the Lagrangian density is a minimum

$$\delta\{\textstyle\iiiint dx\,dy\,dz\,dt\,\mathscr{L}(\mathbf{E},\mathbf{H})\}=0. \tag{8}$$

We now make use of the paraxial approximation:

$$|\partial e_s(\mathbf{r})/\partial z|\ll k_s|e_s(\mathbf{r})| \tag{9}$$

and the boundary conditions:

$$\lim_{x,y\to\infty} e_s(\mathbf{r})=0; \quad e_s(\mathbf{r})|_{z=0}\ \text{given} \tag{10}$$

so that the Lagrangian becomes, upon substitution of (2–7) into (1):

$$\begin{aligned}L=(-1/4\mu_0\omega_s^2)&\textstyle\iiiint dx\,dy\,dz\,dt\\ &\times[|\partial e_s/\partial x|^2+|\partial e_s/\partial y|^2-2k_s\,\mathrm{Im}\{e_s^*\partial e_s/\partial z\}\\ &-(3/2)(k_s^2/n_s^2)\chi^{(R)}|e_p e_s|^2].\end{aligned} \tag{11}$$

Here it is assumed that the integrals over z and t of the Lagrangian density extend over many cycles of the Stokes field. Therefore the rapidly varying components in z and t do not contribute to (1) and have been omitted from (11). The Euler-Lagrange equation for the above is just the paraxial ray equation:

$$\begin{aligned}&[\partial^2/\partial x^2+\partial^2/\partial y^2-2ik_s\partial/\partial z\\ &\quad+(3/2)(k_s^2/n_s^2)\chi^{(R)}|e_p|^2]e_s=0\end{aligned} \tag{12}$$

a full solution of which has been sought by Perry et al. [5, 6], and more recently by Gavrielides and Peterson [7] who have also taken into account depletion of the pump beam. Their approach was to pose (12) as an eigenvector problem in the Hilbert space of Gauss-Laguerre functions which are the TEM free-space modes. The associated eigenvalues represent the growth of the Stokes beam on propagation through the gain medium. For the particular case $k_s=k_p$, Perry et al. gave their results for the variation of the three largest eigenvalues with the pump power. Although theirs is an exact (numerical) solution of (12), an approximate analytic treatment would in some cases be more desirable. For instance, one is generally interested in the component of the Stokes beam that couples into an optimally chosen TEM_{00} beam, whereas the spatial transverse profile of the Stokes beam at the exit of the gain medium is not readily recoverable from the Gauss-Laguerre eigenvectors.

The following treatment therefore models the Stokes field as a Gaussian beam throughout the medium, the parameters of which are chosen to minimise (11). Our approximation consists of ignoring the coupling between this and higher-order modes, although it will seen that this approach becomes exact either when the pump power is sufficiently large or sufficiently small. We therefore retain the Lagrange formulation and substitute into (11) a Stokes field of the form

$$e_s(\mathbf{r})=A(z)\exp[-iQ(z)r^2/2]. \tag{13}$$

The amplitude $A(z)$ and beam parameter $Q(z)$ are now chosen so that (11) is a minimum. Thus we carry out the transverse integrations, and apply the Euler-Lagrange equations for the variation of $Q^*(z)$ and $A^*(z)$:

$$\frac{3k_s^2\chi^{(R)}|e_{p0}|^2w_{p0}^2}{n_s^2[4+\mathrm{i}(Q-Q^*)w_p^2(z)]^2}=\frac{k_sQ'+Q^2}{(Q-Q^*)^2}, \tag{14}$$

$$k_s\frac{A'}{A}=\frac{|Q|^2+k_sQ'}{(Q-Q^*)}+\frac{3k_s^2\chi^{(R)}|e_{p0}|^2w_{p0}^2(Q-Q^*)}{4n_n^2[4+\mathrm{i}(Q-Q^*)w_p^2(z)]}. \tag{15}$$

Equations (14 and 15) can be recast in terms of the normalised quantities as follows

$$q^2+q'+\frac{\mathrm{i}\tilde{P}_p}{2\kappa^2}\cdot\frac{1}{[1+\xi^2-(\kappa\,\mathrm{Im}\{q\})^{-1}]^2}=0, \tag{16}$$

$$\frac{a'}{a}=\frac{|q|^2+q'}{2\mathrm{i}\,\mathrm{Im}\{q\}}+\frac{\tilde{P}_p}{4\kappa}\cdot\frac{1}{[1+\xi^2-(\kappa\,\mathrm{Im}\{q\})^{-1}]}, \tag{17}$$

where $\tilde{P}_p$ is the normalised pump power:

$$\tilde{P}_p=(3/2)(k_s^2/n_s^2)\,\mathrm{Im}\{\chi^{(R)}\}\,|e_{p0}|^2w_{p0}^2. \tag{18}$$

$q(\xi)$, $a(\xi)$ are, respectively, the new normalised complex beam parameter and amplitude:

$$q(\xi)=w_{p0}^2Q(z)/2\kappa;\quad a(\xi)=A(z), \tag{19}$$

where ξ is the new longitudinal ordinate:

$$\xi=2z/k_pw_{p0}^2 \tag{20}$$

(it is assumed that the origin of the z coordinate is chosen to coincide with the position of the pump focus), and $\kappa=k_s/k_p$ is the ratio of Stokes to pump wave numbers. We have used the definition $\tilde{P}_p$ of [2] apart from the definition of $\chi^{(R)}$ which is that of [11] (the real part is $\chi^{(R)}$ is assumed to be zero). Clearly if $\tilde{P}_p=0$, (16 and 17) reduce to the equations of motion for the spot-size, radius of curvature, and (complex) amplitude of a free-space Gaussian beam. When $\tilde{P}_p \neq 0$ however, these equations can be used both to analyse the results of earlier authors in the domains of low and high pump power, and also provide a more general description for the Stokes field for arbitrary $\tilde{P}_p$; these then are the respective goals of the sections which follow.

2. Solution to the Equations of Motion

2.1. Low Pump Power

We start by considering the limit of low pump power of the solutions of Eqs. (16 and 17). We will first derive the general result for the Stokes amplitude and profile, and then show how this result can be applied to the design of a Raman gain cell.

If the normalised pump power $\tilde{P}_p$ is sufficiently small, the Stokes profile remains almost unchanged from its free-space behaviour

$$q^2+q'=0. \tag{21}$$

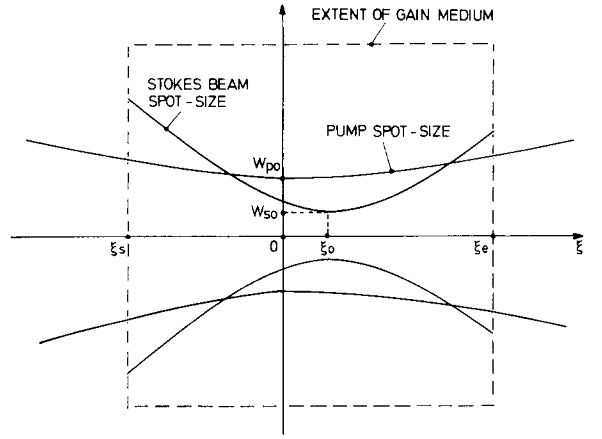

Fig. 1. Configuration of pump and Stokes beam in a gain medium

In terms of normalised quantities, the solution of (21) is

$$q=\mu/[\mu(\xi-\xi_0)+\mathrm{i}] \tag{22}$$

where

$$\xi_0=2f_s/k_pw_{p0}^2, \tag{23}$$

$$\mu=k_pw_{p0}^2/k_sw_{s0}^2. \tag{24}$$

Hence, ξ_0 is the distance of the Stokes focus from the pump focus in units of the pump confocal beam parameter, whilst μ is the ratio of pump to Stokes confocal beam parameter. This general case is depicted in Fig. 1 where the pump and Stokes beams have been enclosed by the gain medium. Of course, calculation of the Stokes field through equations (16 and 17) apply only to the field within the cell. Equally it is tacitly assumed that the finite transverse dimensions of the medium can be ignored.

With the free-space form for the Stokes profile, (17) can easily be solved to give the amplitude of the Stokes field at any point ξ in the gain medium

$$a(\xi)=a(\xi_s)\left[\frac{1-\mathrm{i}\mu(\xi_s-\xi_0)}{1-\mathrm{i}\mu(\xi-\xi_0)}\right]\times\exp\left[\frac{\tilde{P}_p}{4\eta}\left(\tan^{-1}\{[(\kappa+\mu)\xi-\mu\xi_0]/\eta\}-\tan^{-1}\{[(\kappa+\mu)\xi_s-\mu\xi_0]/\eta\}\right)\right], \tag{25}$$

where

$$\eta=[1+\kappa(\mu+\mu^{-1})+\kappa^2+\kappa\mu\xi_0^2]^{1/2}, \tag{26}$$

and $a(\xi_s)$ is the Stokes amplitude at the entrance to the medium. The total power in the Stokes beam can be evaluated from (13, 22, and 25)

$$P_s(\xi)=P_s(\xi_s)\exp\left[\tilde{P}_p/(2\eta)\left(\tan^{-1}\{[(\kappa+\mu)\xi-\mu\xi_0]/\eta\}\right.\right.$$
$$\left.\left.-\tan^{-1}\{[(\kappa+\mu)\xi_s-\mu\xi_0]/\eta\}\right)\right]. \quad (27)$$

Examination of (16) reveals that to first order in $\tilde{P}_p$, the third term also contributes to the gain as described by (17). Thus we find that even for low pump powers, the effect of the pump power on the Stokes beam profile can be significant. However, this component can be shown to be identically zero for the particular initial Stokes profile satisfying: $\xi_0=0$, and $\mu=1$, which is just that the pump and Stokes beams share a focal plane, and have equal confocal parameters. Our motive for ignoring this term is so that we may compare our result with that of earlier workers. Thus Boyd et al. [1] have assumed that the condition $\xi_0=0$ is satisfied, and otherwise obtained the same result using an "overlap integral" method. Equation (27) is also related to the result obtained by Christov and Tomov [4], and allowing for typographical errors, a similar result has been obtained by Trutna and Byer [3]. However, we note that the procedure for the maximisation of the Stokes gain executed by Trutna and Byer is not valid within the framework adopted in this paper. Their choice of an optimal profile, found to be that satisfying $\mu=1$, is a necessary prerequisite for the validity of (27) in the Lagrangian formulation.

Fortunately, the Stokes beam with initial conditions satisfying $\xi_0=0$, $\mu=1$, is exactly the form taken – in the limit of low pump power – by the more general "matched mode" solution of Sect. 2.3. Therefore we will proceed assuming that these conditions are met by the design of the Raman amplifier, so that by virtue of our more general approach, we will then be in a position to determine the validity of the low gain approximation adopted by earlier workers. In this case, the power gain for the Stokes beam is found to be:

$$P_s(\xi)=P_s(\xi_s)\exp[\tilde{P}_p\Theta(\xi;\xi_s)/2(1+\kappa)], \quad (28)$$

where

$$\Theta(\xi;\xi_s)=\tan^{-1}(\xi)-\tan^{-1}(\xi_s). \quad (29)$$

In Sect. 2.3, we will show how this result can be derived from the matched-mode model in the limit of small pump power and will then be in a position to estimate its range of validity.

2.2. High Pump Power

As the pump power is increased, the Stokes profile will deviate from the free-space form given by (22). Hence the extent of the profile is governed by the competing effects of diffraction and gain-focussing determined respectively by the first and third terms of (16). When the pump power is sufficiently great, the effect of gain-focussing is to confine the Stokes spot-size to an area well within that of the "guiding" pump, i.e. in the limit of high pump power we expect $w_p^2(z)\gg w_s^2(z)$. The Stokes spot-size can be defined using (13 and 19), in terms of the normalised variable q; whilst the pump spot-size can be defined using (7 and 20) in terms of the normalised co-ordinate ξ. Thus we may rewrite this condition as

$$1+\xi^2\gg-(\kappa\,\mathrm{Im}\{q\})^{-1}. \quad (30)$$

(If a TEM_{00} Stokes mode exists, then the imaginary part of q must always be negative.) In this limit, (16) becomes

$$q^2+q'+\mathrm{i}\tilde{P}_p/2\kappa^2(1+\xi^2)^2=0. \quad (31)$$

We note that the same result can be obtained for the profile of the field which is a solution to the paraxial ray equation (12) by retaining only the zeroth and quadratic terms in the expansion of $|e_p|^2$ in powers of r^2. Hence this approach is equivalent to that of the parabolic-index profile approximation considered by Cotter et al. [2]. In this paper, however, we proceed to solve for q without the additional approximations made in that work.

Equation (31) is a Ricatti equation, and can be cast as a linear second-order differential equation by making the usual change of variable:

$$q=v^{-1}dv/d\xi \quad (32)$$

also we define

$$\gamma=(1+\mathrm{i}\tilde{P}_p/2\kappa^2)^{1/2} \quad (33)$$

whereupon

$$d^2v/d\xi^2+v(\gamma^2-1)/(1+\xi^2)^2=0. \quad (34)$$

By substitution or otherwise, the solution of (34) can be shown to be

$$v(\xi)=v_0(1+\xi^2)^{1/2}\cos[\gamma\Theta(\xi;\xi_s)+\phi] \quad (35)$$

and v_0, ϕ are (complex) arbitrary constants. The complex parameter q can be recovered using this result and the definition in (32)

$$q(\xi)=\{\xi-\gamma\tan[\gamma\Theta(\xi;\xi_s)+\phi]\}/(1+\xi^2), \quad (36)$$

where now ϕ can be interpreted in terms of the Stokes parameters at the entrance to the gain medium

$$\phi=\tan^{-1}\{[\xi_s-(1+\xi_s^2)q(\xi_s)]/\gamma\}. \quad (37)$$

The amplitude of the Stokes field can now be obtained from (17), where once again we make use of (30)

$$a'/a = -q + \tilde{P}_p/4\kappa(1+\xi^2). \tag{38}$$

Again, recalling the substitution in (33) and the result for $v(\xi)$ in (35), the amplitude can be written down without further calculation:

$$a(\xi) = a(\xi_s)\left(\frac{1+\xi_s^2}{1+\xi^2}\right)^{1/2} \frac{\cos(\phi)}{\cos(\gamma\Theta(\xi;\xi_s)+\phi)} \times \exp[\tilde{P}_p\Theta(\xi;\xi_s)/4\kappa]. \tag{39}$$

Consider now the initiation of the stimulated process from spontaneous scattering at $\xi = \xi_s$. If the initial field is a Gaussian beam with zero spot-size and zero radius of curvature, then we would have $q(\xi_s) \to \infty - i\infty$. It is easily seen that after a short distance into the gain medium, the complex parameter q from (36) obeys

$$q(\xi) \simeq (\xi - i\gamma)/(1+\xi^2) \tag{40}$$

and the Stokes amplitude is

$$a(\xi) \simeq 2ia(\xi_s)\left(\frac{1+\xi_s^2}{1+\xi^2}\right)^{1/2} \exp[(\tilde{P}_p/4\kappa + i\gamma)\Theta(\xi;\xi_s)]. \tag{41}$$

In the derivation of (40 and 41), use has been made of the constraint

$$|\exp[-2i\gamma\Theta(\xi;\xi_s)]| \gg 1; \quad \xi_s \leqq \xi \leqq \xi_e. \tag{42}$$

This is a simplifying assumption designed to ensure that the cosine terms in (39) effectively collapse into the dominant exponential component. The value of ξ for which (42) becomes true depends on the magnitude of the gain: the higher the gain, the earlier will this constraint be satisfied and therefore will q approach the particular form given in (40).

Defining the real and imaginary parts of Q in terms of the spot-size and radius of curvature (see, for instance, [12])

$$Q = k_s/R_s(z) - 2i/w_s^2(z) \tag{43}$$

then we find that (40) implies that the Stokes beam has a radius of curvature

$$R_s(z) = k_p w_{p0}^2(1+\xi^2)/2(\xi + \mathrm{Im}\{\gamma\}) \tag{44}$$

and spot-size given by

$$w_s^2(z) = w_p^2(z)/\kappa\,\mathrm{Re}\{\gamma\}. \tag{45}$$

Hence the Stokes field is a Gaussian beam with propagation characteristics similar to that of a free-space beam, but with a distorted phase front, and a spot-size that is everywhere narrower than its free-space equivalent.

Equations (40 and 41) describe the "matched mode" behaviour of the Stokes field in that the complex parameter $q(\xi)$ and amplitude $a(\xi)$ have become independent of the initial parameter $q(\xi_s)$. This is a generalisation of a concept first introduced in this context by Cotter et al. [2]. The magnitude of the pump power, through the left hand side of (42), is seen to determine how quickly the initial Stokes profile tends towards the matched mode profile. In fact, if instead of an initial field with zero spot-size and zero radius of curvature, the initial parameter is made to satisfy the matched mode condition at $\xi = \xi_s$

$$q(\xi_s) \simeq (\xi_s - i\gamma)/(1+\xi_s^2) \tag{46}$$

then the $q(\xi)$ remains unchanged from its matched mode value (40) throughout the medium.

These results can be compared with those of [2] by taking the high pump power limit for the complex parameter γ defined in (33). Under these conditions, the Stokes power is

$$P_s(\xi) \simeq 4P_s(\xi_s)\exp[(\tilde{P}_p/2 - \sqrt{\tilde{P}_p})\Theta(\xi;\xi_s)/\kappa]. \tag{47}$$

Therefore, the results of [2] represent the high pump power limit of the matched mode solution. Note, however, by virtue of the constraint on the size of the Stokes beam, expressed approximately through (32), that this result is true only when $\tilde{P}_p \gg 4$. Thus the explanation based on this result which was advanced by Cotter for the behaviour of the Stokes beam at low pump power is spurious. Note also that (47) describes a Stokes power similar in form to that obtained from the low pump power calculation of the previous section: the first term in the exponent is greater by a factor $(1+\kappa)/\kappa$, whilst the additional second term represents a reduction in gain due to the increased diffraction of the Stokes field in the presence of gain-focussing.

2.3. *Matched Mode*

Following the discussion in the previous section, we now seek an exact matched mode solution to the equations of motion (16 and 17) without making the parabolic index profile approximation. The result will then be an analytic description for the Stokes field that will be valid simultaneously under conditions of low pump power as for example in a multi-pass Raman gain cell, and conditions of high pump power likely to be encountered in a single-pass Raman generator. In either case, the matched mode condition may be arrived at through one of two routes:

(a) An initially unmatched mode perturbed by the gain medium to a point where the spot size and radius of curvature have converged upon that of the matched mode. From the previous section, we find this condition will generally be satisfied if

$$|\exp[-2i\gamma\Theta(\xi;\xi_s)]| \gg 1; \quad \xi_s \leqq \xi \leqq \xi_e. \tag{42}$$

(b) An injected field which is a Gaussian beam with spot-size and radius chosen to satisfy the matched mode condition at $\xi = \xi_s$.

The (exact) matched mode solution to (16) may be derived from a substitution of the form

$$q(\xi) = (c_0 + c_1\xi)/(1+\xi^2); \quad \mathrm{Im}\{c_0\} < 0. \tag{48}$$

Upon equating equal powers of ξ and setting $c_0 = \alpha - i\beta$ (where α, β are real) we obtain

$$c_1 = 1, \tag{49}$$

$$\alpha = (\beta^2 - 1)^{1/2}, \tag{50}$$

$$\beta\tilde{P}_p = 4(\beta^2-1)^{1/2}(1+\kappa\beta)^2. \tag{51}$$

Equation (51) gives that β is the solution of a sixth order polynomial with co-efficients which are simple functions of $\tilde{P}_p$ and κ. A series and graphical solution is given in the next section, for now we note that $\beta = 1$ at $\tilde{P}_p = 0$, and β increases as $\tilde{P}_p^{1/2}$ when $\tilde{P}_p$ is large. With the substitution (48), the matched mode amplitude is found from (17) to satisfy

$$a'/a = [\tilde{P}_p\beta/4(1+\kappa\beta) - \xi + i\beta]/(1+\xi^2). \tag{52}$$

Therefore the Stokes power can be written

$$P_s(\xi) = P_s(\xi_s)\exp[G(\tilde{P}_p, \Theta, \kappa)], \tag{53}$$

where $G(\tilde{P}_p, \Theta, \kappa)$ is the matched-mode exponential power gain

$$G(\tilde{P}_p, \Theta, \kappa) = \tilde{P}_p\Theta\beta/2(1+\kappa\beta) \tag{54}$$

and the matched-mode complex parameter q is

$$q(\xi) = [(\beta^2-1)^{1/2} + \xi - i\beta]/(1+\xi^2) \tag{55}$$

and β is given by the solution of (51).

For the general matched-mode result, it may be of interest to know the spot-size and radius of curvature at any point ξ in the medium. Comparison of (40) with (48) reveals that the substitution $\gamma \to ic_0$ renders the two forms identical, so that we can use results (44 and 45) to obtain in the matched-mode domain

$$R_s(z) = k_p w_{p0}^2(1+\xi^2)/2[\xi + (\beta^2-1)^{1/2}], \tag{56}$$

$$w_s^2(z) = w_p^2(z)/\kappa\beta, \tag{57}$$

where again β is given by the solution of (51). The radius and spot size at the end of the gain medium can be found simply by substituting $\xi = \xi_e$ into (56 and 57), respectively. It is clear from these results and the fact that $\beta \geqq 1$, that the radius of curvature and the spot size of the matched mode are smaller than that of the "equivalent" free-space mode (which has $\mu = 1$ and shares a focal plane with the pump beam).

3. Discussion

The matched-mode power gain and beam profile are modified by the functional dependence of β on the $\tilde{P}_p$ and κ. The latter is given by the solution of (51) which can be developed as a series in powers of $\tilde{P}_p$ (with coefficients which are functions of κ). When the pump power is low, then it is correct to develop β in increasing powers of $\tilde{P}_p^2$ whereupon we find

$$\beta = 1 + \tilde{P}_p^2/32(1+\kappa)^4 + \tilde{P}_p^4(3-5\kappa)/2048(1+\kappa)^9 + \ldots \tag{58}$$

and from (54), it is easy to show that the Stokes power gain is:

$$G = \Theta[\tilde{P}_p/2(1+\kappa) + \tilde{P}_p^3/64(1+\kappa)^6 + \tilde{P}_p^5(3-7\kappa)/4096(1+\kappa)^{11} + \ldots]. \tag{59}$$

The first term is just the gain for the low pump power approximation of Sect. 2.1 for which the second and higher terms can therefore be regarded as corrections. We can estimate the range of validity of that approximation by comparing the first and second terms to give

$$\tilde{P}_p \ll (2(1+\kappa))^{5/2}. \tag{60}$$

The same technique can be applied to the solution of (51) when the pump power is large. In this case it is correct to develop β in decreasing powers of $\tilde{P}_p^{1/2}$ whereupon we find

$$\beta = \tilde{P}_p^{1/2}/2\kappa - 1/\kappa + \kappa\tilde{P}_p^{-1/2}/2 + 2\kappa\tilde{P}_p^{-1} + \ldots \tag{61}$$

and

$$G = \Theta[\tilde{P}_p/2\kappa - \tilde{P}_p^{1/2}/\kappa + \kappa P_p^{-1/2} + \ldots]. \tag{62}$$

The first two terms represent the result for the high pump power approximation of Sect. 2.2, for which, by comparison with the third term, the necessary constraint is found to be

$$\tilde{P}_p^3 \gg 4(\tilde{P}_p + \kappa^2)^2. \tag{63}$$

This constraint, which replaces that of (30) in Sect. 2.2, is satisfied if $\tilde{P}_p \gg 5.6$ when $\kappa = 1$, and at the other extreme, if $\tilde{P}_p \gg 4$ when $\kappa = 0$.

We note in passing that the gain-focussed Stokes beam becomes ever more confined with increasing pump power and therefore can expect the parabolic index profile approximation discussed in Sect. 2.2 to give increasingly accurate results. Thus the coupling between modes will eventually vanish and the high gain limit given by (53–55) with $\beta = \tilde{P}_p^{1/2}/2\kappa$ will give the exact solution to (12). Further it is recalled that the result first obtained by Trutna and Byer [3] is effectively that of a first-order perturbation theory (in the pump intensity) applied to a Stokes field expansion in free-space TEM_{nm} modes. Hence for sufficiently low pump powers, the limit given by $\beta = 1$ will also give an exact solution to (12).

We will now illustrate graphically the predictions of the matched-mode model assuming for definiteness that the pump focussing conditions are such that $\Theta(\xi; \xi_s) \simeq \pi$, and therefore that the Stokes exponential power gain is given by $G(\tilde{P}_p, \pi, \kappa)$. Equation (51) has been solved for β numerically, and a plot of β versus $\tilde{P}_p$ for various values of κ is given in Fig. 2. These results can be used to find the matched mode exponential gain in (54), and the spot-size and radius of curvature in (56 and 57). In Fig. 3 we compare the matched mode gain with the gains for the high and low pump power models predicted, respectively, by the first term of (59), and the first and second terms of (62). As expected, it is seen that the limiting cases are modelled in a satisfactory way as $\tilde{P}_p \to 0$ and $\tilde{P}_p \to \infty$, respectively. For the chosen value of $\kappa = 1$, we observe that the predictions of the low and high-gain approximations are equal at a pump power $\tilde{P}_p = 16$ (the high pump power solution thereafter being closer than the solution for low pump power to that of the matched mode). In general, it can be shown that the two predictions will be the same when $\tilde{P}_p = 4(1+\kappa)^2$.

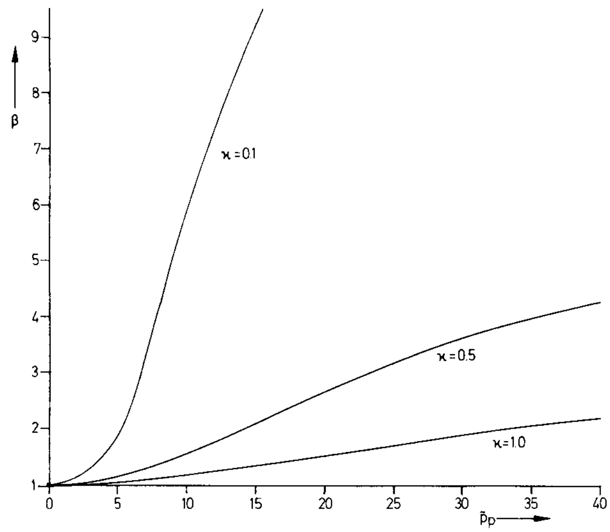

Fig. 2. Dimensionless parameter β versus normalised pump power $\tilde{P}_p$

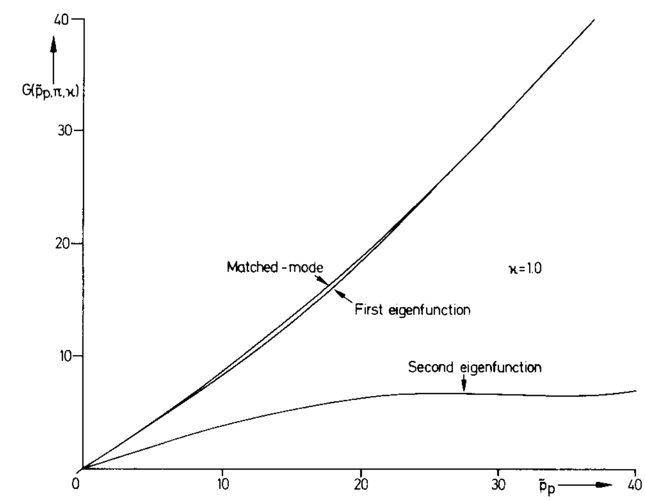

Fig. 4. Stokes exponential gain as predicted by matched-mode theory and that of first and second eigenfunctions of Perry et al.

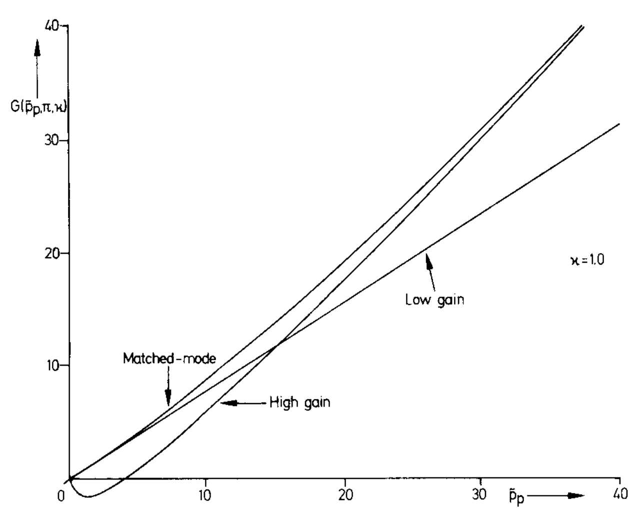

Fig. 3. Stokes exponential gain as predicted by matched-mode, low-gain and high-gain theory

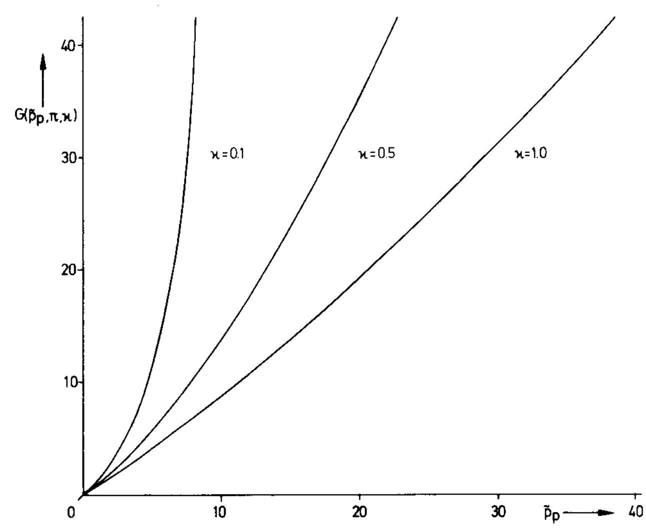

Fig. 5. Stokes exponential gain as predicted by matched-mode theory for $\kappa = 0.1$, 0.5, and 1.0

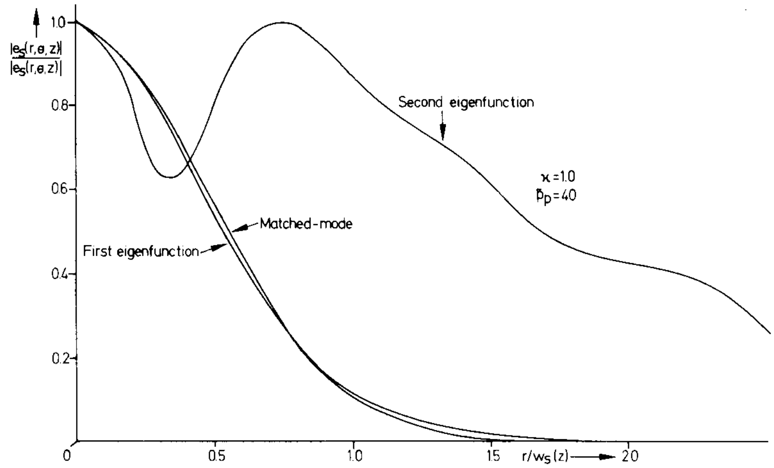

Fig. 6. Stokes profile as predicted by matched-mode theory and that of first and second eigenfunctions of Perry et al.

It is also of interest to compare these results with those obtained by Perry et al. [5]. First it is necessary to make explicit the connection between the symbols used in their work, and those adopted in this paper. Table 1 provides a summary of the pertinent relationships:

Table 1

Description	Perry et al.	This paper
Dimensionless parameter	μ	$\kappa/(1+\kappa)$
Gain coefficient	G_p	$\tilde{P}_p/4$
Real part of eigenvalue	$\mathrm{Re}\{\lambda\}$	$\tilde{P}_p\beta/4(\kappa\beta+1)$
Normalised gain	$\mathrm{Re}\{\lambda\}/G_p$	$\beta/(\kappa\beta+1)$

In Fig. 4 we have used these relationships to compare the matched-mode gain with the gains predicted in [5] (at $\kappa=1$) for the first and second (rotationally symmetric) eigensolutions to the paraxial ray equation (12). Clearly the matched-mode gain is consistently close to the gain of the first eigensolution and the excellent correspondence between these results lends support to our model. Encouraged by this comparison, we present in Fig. 5 the matched mode gain for various values of κ found by applying the solution of (51–54).

It is possible to further test the accuracy of our supposition that the lowest-order mode is essentially a Gaussian beam by comparing the matched mode profile with that predicted by the numerical results of [5]. With reference to Fig. 6, we have used the normalised coordinate $r/w_s(z)=r\sqrt{\kappa\beta}/w_p(z)$, and find that once again, at least for the values $\tilde{P}_p=40$ and $\kappa=1$, there is very good agreement between the results.

4. Summary

In this paper we have presented an analytic model for the evolution of a Stokes field in a Raman active medium excited by a focussed pump beam. We have shown that our results are valid throughout a wide range of values for the pump power, and that in the limits of high and low pump power, they reproduce the results of earlier workers. We have therefore been able to identify constraints which in this context define the domains of high and low pump power. Excellent agreement has been obtained in comparison with an earlier numerical treatment.

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