On the Equivalence of Finite Difference and Finite Element Formulations in Magnetic Field Analysis

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***Abstract*—Numerical 3D formulations using scalar Ω and vector *A* potentials are examined for magnetic fields, with emphasis on the finite difference (FDM) and finite element (FEM) methods using nodal and facet elements. It is shown that for hexahedral elements the FDM equations may be presented in a form similar to the FEM equations; to accomplish this the coefficients defining volume integrals in FEM need to be expressed in an approximate manner, while the nodes in FDM require supplementary association with middle points of edges, facets and volumes.**

***Index Terms*—magnetic fields, finite difference methods, finite element analysis, functional analysis, education.**

I. Introduction

THE OLDEST historically numerical routine for magnetic field modelling is the finite difference method (FDM); in its 2D guise it was likened to a classical finite element method (FEM) using nodal formulation [1]-[3] – those publications instigated some spirited debate at the time. In particular, the possibility to derive FDM equations from an energy functional similar to FEM was pointed out in the discussion following [3]. In the classical FEM approach shape functions are employed; in the FDM the functional results from the definition of finite differences, where the average energy density in an element is a weighted average of values at nodes or at points between the nodes. Consequently, the final equation differs from a typical FDM formulation only in the description of the average flux density. The aforementioned papers, however, considered only classical nodal FEM equations and did not include the edge or facet formulations typical for 3D.

In previous publications the authors of this paper have focused on the ways analogies could be established between different methods. In particular, it has been shown that through suitable assumptions and approximations the FEM equations could be made to be identical to those obtained from the Finite Integration Technique (FIT) or equivalent reluctance networks [4], [5]. Analogies between edge formulations in FEM and FDM were also established when a magnetic vector potential ***A*** was employed [6]. The purpose of this paper is to extend the treatment to edge, facet and volume formulations of the FEM.

The following discussion will apply to formulations using the magnetic vector potential ***A*** for enforced current densities ***J*** or the electric vector potential ***T*** for magnetisation and conduction currents, as well as the magnetic scalar potential Ω when the distribution of the vector potential ***T*** is imposed. It has been recognised that in the classical FDM scheme the field quantities are associated with points (nodes) whereas in FEM may also be related to edges, facets or volumes.

II. Nodal, Edge, Facet and Volume Values in FDM

Consider the 8-node element (P_1 to P_8) depicted in Fig. 1. In addition to those nodes, ancillary points have been specified associated with the element's edges, facets and volume, positioned in the middle of the respective geometrical feature. At points in the middle of each edge the values of ***A***, ***T***, ***H*** and gradΩ are defined, whereas in the middle of each facet also the values of ***B*** and ***J***. A simple relationship exists between the edge value $\varphi_{Ei,j}$ of a vector ***E*** (where ***E*** = ***A***, ***T***, ***H*** or gradΩ) for the edge P_iP_j of the length Δu (where $u = x$, y or z) and the value $E_{ui,j}$ of the relevant u component of ***E*** at the point $Q_{i,j}$, namely $\varphi_{Ei,j} = \Delta u E_{ui,j}$. From this it is clear that the finite difference defined in FDM as $(\Omega_j - \Omega_i) \approx \Delta u\, \mathrm{grad}\Omega(Q_{i,j})$ relates to the edge value of gradΩ for the edge P_iP_j.

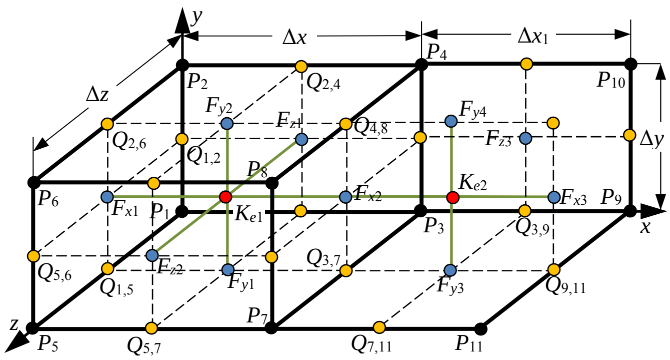

Fig. 1. Characteristic points for a hexahedral element.

The facet value of a vector ***S*** (***S*** = ***J***, ***B***, ***H***) for a facet with the middle point F_{uk} ($u = x, y, z$; $k = 1, 2$) may be expressed as a product of the area of that facet and the u component of ***S*** at the point F_{uk}. For example, the relationship between the facet value f_{y1} of the vector ***S*** and S_{y1}, the value of a component of the vector S_y at point F_{y1}, may be written as $f_{y1} = \Delta x \Delta z S_{y1}$. In dual finite element formulation a mesh has branches connecting the middle points K_{ei} of adjacent volumes cutting a facet through its middle point F_{uk} [7], [8]. Following [2], we will distinguish between the 'edge network' (EN) with nodes P_i and the 'facet network' (FN) with nodes at K_{ei}. In the case of FN, the volume value of potential Ω in an element is specified by the product of the potential Ω_{ei} in node K_{ei} and the element volume. Moreover, the edge value of ***S*** for the edge $K_{ei}F_{uk}$ is given by the volume integral of the product of ***S*** and the interpolating function of the facet element for the facet with its middle point at F_{uk}. It can be shown that the edge value of gradΩ is then equal to the difference between the average value Ω_{ei} of Ω in the volume and the average value Ω_{Fuk} of Ω for the facet F_{uk}.

III. NETWORK REPRESENTATION OF FEM AND FDM

Analogies between FEM and equivalent magnetic networks are helpful. FEM schemes relying on scalar potential Ω and nodal elements are related to nodal equations of the permeance network EN with nodes P_i, whereas using vector potential $\boldsymbol{A}$ and edge formulation is equivalent to the loop equations of the permeance network FN with nodes at K_{ei} and loops around the edge P_iP_j [4]. The loop fluxes φ in FN represent the edge values of $\boldsymbol{A}$. In magnetic networks equivalent to FEM, couplings exist between branches of an element, i.e. mutual permeances $\Lambda_{ij,pq}$ between branches P_iP_j and P_qP_p in EN, or mutual reluctances $R_{\mu uk,r}$ between branches $K_{ei}S_{uk}$ and $K_{ei}S_{ur}$ in FN [4]. The relevant parameters may be calculated from

$$\Lambda_{ij,pq} = \iiint_{V_e} \boldsymbol{w}_{eij}^T \mu \boldsymbol{w}_{epq} \mathrm{d}\boldsymbol{v}\,, \qquad R_{\mu uk,r} = \iiint_{V_e} \boldsymbol{w}_{fk}^T \mu^{-1} \boldsymbol{w}_{fr} \mathrm{d}\boldsymbol{v} \qquad (1\text{ a, b})$$

where $\boldsymbol{w}_{eij}$ and $\boldsymbol{w}_{epq}$ are the interpolating functions of the edge element for the edge P_iP_j, P_pP_q, whereas $\boldsymbol{w}_{fk}$ and $\boldsymbol{w}_{fr}$ are the interpolating functions for the k*th* and r*th* facet of the facet element. The integrals are usually approximated, for example as described in [5], where the approximation replaces the integrals by a product of the volume V_e and the average value of the integrand at nodes P_i. This results in the mesh equations for the hexahedral (cuboid) elements being free of mutual terms, whereas self permeances or reluctances are described through simple relationships, e.g. for the branch P_1P_2 the permeance $\Lambda_{12,12}=\mu\Delta x\Delta z/(4\Delta y)$, while for the branch K_eF_{y2} the reluctance $R_{\mu y2,2}=\Delta y/(2\mu\Delta x\Delta z)$. The absence of mutual terms makes the inversion of the branch parameters matrix of the equivalent meshes a much easier task. Thus the field distribution given by the edge values of $\boldsymbol{A}$ in FN may be found via a process of solving the equations for nodal potentials Ω_{ei}. Similarly, the solution of equations describing the distribution of Ω at nodes P_i of EN may be converted to a task of finding edge values of $\boldsymbol{A}$ for the edge $K_{ei}K_{ej}$. In this last transformation it is recognised that the edge value of $\boldsymbol{A}$ for $K_{ei}F_{uk}$ is related to the integral of the product $\boldsymbol{w}_{fk}\boldsymbol{A}$ and that $\boldsymbol{A}$ may be expressed in terms of the values for P_iP_j. In the language of circuit theory these transformations result in loop equations for FN being replaced by nodal equations and the nodal equations for EN by loop equations for loops assigned to element facets (see Fig. 2).

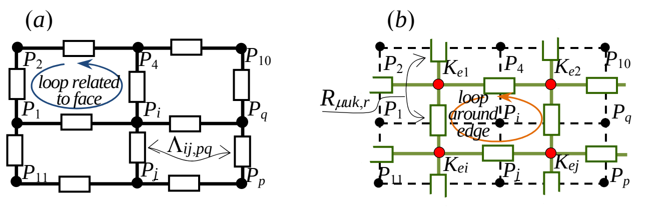

Fig.2. Parts of equivalent magnetic networks: (a) edge - EN and (b) facet - FN

Similar models may be derived from FDM; if working with magnetic vector potential the procedures derived in [6] should be followed, thus assume the product of the u component of $\boldsymbol{A}$ at point $Q_{i,j}$ and the edge length Δu in the direction of u to be the unknown, bearing in mind that the reluctivity at the centre F_{ui} of a facet is a weighted average of the volume values in elements attached to the facet. When using the scalar potential it must be remembered that the permeability $\mu(Q_{i,j})$ at the centre of the edge P_iP_j is a weighted average for the four adjacent elements sharing the edge. When the energy functional is set up then inside an element $B^2=\Sigma_u(\Sigma_{i=1,2}(B_{ui})^2)/2$, $H^2=\Sigma_u(\Sigma_{i,j}(H_{uij})^2)/4$, where $B_{ui}=B_u(F_{ui})$, $H_{uij}=H_u(Q_{i,j})$, $u=x,y,z$.

IV. REPRESENTATION OF SOURCES

Sources may be described in two ways, either in terms of the imposed (prescribed) current density $\boldsymbol{J}$ using facet elements, or by working with edge elements and applying imposed (in the case of permanent magnets) or derived (e.g. from $\boldsymbol{J}$=curl$\boldsymbol{T}$) distributions of electric vector potential $\boldsymbol{T}$ or $\boldsymbol{T}_0$. The former yields the loop *mmf*s, it is therefore only suitable for mesh methods, e.g. using the magnetic vector potential $\boldsymbol{A}$, whereas the latter is more universal as from the edge values of $\boldsymbol{T}$ or $\boldsymbol{T}_0$ branch *mmf*s may be established thus making the description applicable to both nodal and loop methods, i.e. appropriate for derivations using either Ω or $\boldsymbol{A}$. Modern FEM formulations tend to use the latter description; it should be noted, however, that well before the advent of edge element formulation a version of this approach was already common in FDM. As an example, a popular phrase referred to current linkage distribution created by electrical machine winding. The analogy between FDM and FEM therefore extends to field sources.

V. CONCLUSION

The equivalence between finite difference (FDM) and finite element (FEM) formulations, under certain assumptions, has been demonstrated. For rectangular parallelepiped (cuboids), when approximations are applied to integrals arising from the FEM formulation, equations suitable to FDM emerge for points associated not only with element nodes but also with edges, facets and volumes. The resulting equations are identical and the analogy also embraces the representation of field sources. Further correlations in terms of forces and torques and for non-linear systems will be explored in the full version.

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